

Core percolation on complex networks

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As a fundamental structural transition in complex networks, core percolation is related to a wide range of important problems. Yet, previous theoretical studies of core percolation have been focusing on the classical Erdős-Rényi random networks with Poisson degree distribution, which are quite unlike many real-world networks with scale-free or fat-tailed degree distributions. Here we show that core percolation can be analytically studied for complex networks with arbitrary degree distributions. We derive the condition for core percolation and find that purely scale-free networks have no core for any degree exponents. We show that for undirected networks if core percolation occurs then it is always continuous while for directed networks it becomes discontinuous when the in- and out-degree distributions are different. We also apply our theory to real-world directed networks and find, surprisingly, that they often have much larger core sizes as compared to random models. These findings would help us better understand the interesting interplay between the structural and dynamical properties of complex networks.

Network science has emerged as a prominent field in complex system research, which provides us a novel perspective to better understand complexity[1–3]. In the last decade considerable advances about structural and dynamical properties of complex networks have been made[4–6]. Among them, structural transitions in networks were extensively studied due to their big impacts on numerous dynamical processes on networks. Particularly interesting are the emergence of a giant connected component[7–10], k -core percolation[11–13], k -clique percolation[14, 15], and explosive percolation[16–18]. These structural transitions affect many properties of networks, e.g. robustness and resilience to breakdowns[9, 19, 20], cascading failure in interdependent networks[21–24],

epidemic and information spreading on socio-technical systems[3, 25, 26]. Recent work on network controllability reveals another interesting interplay between the structural and dynamical properties of complex networks[27–29]. It was found that the robustness of network controllability is closely related to the presence of the *core* in the network[27, 30]. Actually, core percolation has also been related to many other interesting problems, including conductor-insulator transitions[31, 32] and some classical combinatorial optimization problems, e.g. maximum matching[33–35] and vertex cover[36–38].

The core of a undirected network is defined as a spanned subgraph which remains in the network after the following greedy leaf removal (GLR) procedure[32, 33]: As long as the network has leaves, i.e. nodes of degree 1, choose an arbitrary leaf v_1 , and its neighbor v_2 , and remove them together with all the edges incident with v_2 . Finally, we remove all isolated nodes. It can be proven that the resulting graph is independent of the order of removals[32]. Note that the core described above is fundamentally different from the k -core of a network. The latter is defined to be the maximal subgraph having minimum node degree of at least k , which can be obtained by iteratively removing nodes of degree less than k . Apparently, the GLR procedure described above is more destructive than the node removal procedure used to obtain the 2-core (see Fig.1a). In studying the robustness of controllability for general directed networks, the GLR procedure has been extended to calculate the core of directed networks[27]. We first transform a directed network \mathcal{G} to its bipartite graph representation \mathcal{B} by splitting each node v into two nodes v^+ (upper) and v^- (lower), and we connect v_1^+ to v_2^- in \mathcal{B} if there is a link $(v_1 \rightarrow v_2)$ in \mathcal{G} . The core of a directed network \mathcal{G} can then be defined as the core of its corresponding bipartite graph \mathcal{B} obtained by applying GLR to \mathcal{B} as if \mathcal{B} is a unipartite undirected network.

One can easily tell whether the core exists in two very special cases: (1) If a network has no cycles, i.e. a tree or a forest (a disjoint union of trees), then eventually all nodes will be removed, hence no core. For example, the Barabási-Albert (BA) model with parameter $m = 1$ yields a tree network, hence no core exists. (2) If a network has no leaf nodes, e.g. regular graphs with all nodes having the same degree $k > 1$ or the networks generated by the BA model with $m > 1$, then the GLR procedure will not even be initiated, hence all the nodes belong to the core.

Except those two special cases, no general rules have been proposed to predict the existence of the core for an arbitrarily complex network. Previous theoretical studies focused on undirected Erdős-Rényi (ER) random graph. It has been show that for mean degree $c \leq e = 2.7182818\dots$, the core is small (zero asymptotically), whereas for $c > e$ the core covers a finite fraction of all the nodes[32, 33, 39]. In other words, core percolation occurs at the critical point $c^* = e$. More

interestingly, it has been suggested that in ER random graph core percolation coincides with the changes of the solution-space structure of the vertex cover problem[36, 38, 40], which is one of the basic NP-complete optimization problems[41]. Also, for $c \leq e$ the typical running time of an algorithm for finding the minimum vertex cover is polynomial[32, 36], while for $c > e$, one needs typically exponential running time[42]. Hence, core percolation also coincides with an “easy-hard transition” of the typical computational complexity[38, 40].

Despite the results on undirected ER random networks and the importance of understanding the intriguing interplay between core percolation and other problems, we lack a systematic study and a general theory of core percolation for both undirected and directed random networks with arbitrary degree distributions.

I. ANALYTICAL FRAMEWORK

We propose the following analytical framework to study core percolation on random networks with arbitrary degree distributions. We first categorize the nodes according to how they can be removed during the GLR procedure. We define the following categories: (1) α -removable: nodes that can become isolated (e.g. v_1 and v_2 in Fig.1b); (2) β -removable: nodes that can become a neighbor of a leaf (e.g. v_3 and v_5 in Fig.1b); (3) non-removable: nodes that cannot be removed and hence belong to the core (e.g. v_6, v_7 and v_8 in Fig.1b). While the core is independent of the order the leaves are removed[32], the specific way a node is removed may depend on this order, but it can be proven that no node can be both α -removable and β -removable at the same time. Now we consider an uncorrelated random network with arbitrary degree distribution $P(k)$ [10, 43]. Assuming that in each removable category the removal of a random node can be made locally, we can determine the category of a node v in a network \mathcal{G} by the categories of its neighbors in $\mathcal{G} \setminus v$, i.e. the subgraph of \mathcal{G} with node v and all its edges removed, using the following rules: (1) α -removable: all neighbors are β -removable; (2) β -removable: at least one neighbor is α -removable; (3) non-removable: no neighbor is α -removable, and at least two neighbors are not β -removable.

Let α and β denote the probability that a random neighbor of a random node v in a network \mathcal{G} is α -removable and β -removable in $\mathcal{G} \setminus v$, respectively. We can derive two self-consistent equations

about α and β

$$\alpha = \sum_{k=1}^{\infty} Q(k) \beta^{k-1} = A(1 - \beta), \quad (1)$$

$$1 - \beta = \sum_{k=1}^{\infty} Q(k) (1 - \alpha)^{k-1} = A(\alpha) \quad (2)$$

where $Q(k) \equiv kP(k)/c$ is the degree distribution for the node at a random end of a randomly chosen edge, $c \equiv \sum_{k=0}^{\infty} kP(k)$ is the mean degree, and $A(x) \equiv \sum_{k=0}^{\infty} Q(k+1)(1-x)^k$. These two equations indicate that α satisfies $x = A(A(x))$. It can be shown that α is the smallest fixpoint of $A(A(x))$, i.e. the smallest root of the function $f(x) \equiv A(A(x)) - x$.

The expected fraction of non-removable nodes, i.e. the normalized core size ($n_{\text{core}} \equiv N_{\text{core}}/N$), can then be calculated:

$$n_{\text{core}} = \sum_{k=0}^{\infty} P(k) \sum_{s=2}^k \binom{k}{s} \beta^{k-s} (1 - \beta - \alpha)^s, \quad (3)$$

which can be simplified in terms of $G(x) \equiv \sum_{k=0}^{\infty} P(k)x^k$, i.e. the generating function of the degree distribution $P(k)$. The final result is given by

$$n_{\text{core}} = G(1 - \alpha) - G(\beta) - c(1 - \beta - \alpha)\alpha. \quad (4)$$

For Erdős-Rényi random networks, $G(x) = e^{-c(1-x)} = A(1-x)$, Eq.4 can be further simplified as $n_{\text{core}} = (1 - \beta - \alpha)(1 - c\alpha)$, confirming previous results[32, 39].

The normalized number of edges in the core ($l_{\text{core}} \equiv L_{\text{core}}/N$) can also be calculated in terms of α and β . Consider a uniform random edge, which remains in the core if and only if both of its endpoints are non-removable without removing the edge. The probability of one endpoint being non-removable without removing the edge is $1 - \alpha - \beta$, and for the two endpoints the probabilities are independent. Therefore, the expected normalized number of edges in the core is

$$l_{\text{core}} = \frac{c}{2} (1 - \alpha - \beta)^2. \quad (5)$$

with $c/2 = L/N$ the normalized number of edges in the network. Clearly, both $n_{\text{core}} > 0$ and $l_{\text{core}} > 0$ if and only if $1 - \beta - \alpha > 0$.

Now we consider directed networks \mathcal{G} with given in- and out-degree distributions, denoted by $P^-(k)$ and $P^+(k)$, respectively. Let c denote the mean degree of each partition in the bipartite graph representation \mathcal{B} of the directed network \mathcal{G} , i.e. the mean in-degree (or out-degree) of \mathcal{G} . Define $Q^\pm(k) \equiv kP^\pm(k)/c$, which is the degree distribution of the upper or lower end, respectively,

of a random edge in \mathcal{B} . Define $A^\pm(x) \equiv \sum_{k=0}^{\infty} Q^\pm(k+1)(1-x)^k$. Then the same argument as we used in the undirected case gives that

$$\alpha^\pm = A^\pm(1 - \beta^\mp), \quad (6)$$

$$1 - \beta^\pm = A^\pm(\alpha^\mp) \quad (7)$$

and α^\pm is the smallest fixpoint of $A^\pm(A^\mp(x))$. Now we can calculate the size of the core for each partition in \mathcal{B} as

$$n_{\text{core}}^\pm = \sum_{k=0}^{\infty} P^\pm(k) \sum_{s=2}^k \binom{k}{s} (\beta^\mp)^{k-s} (1 - \beta^\mp - \alpha^\mp)^s \quad (8)$$

and we define the size of the core in the directed network \mathcal{G} as

$$n_{\text{core}} = (n_{\text{core}}^+ + n_{\text{core}}^-)/2. \quad (9)$$

The normalized number of edges in the core can also be calculated

$$l_{\text{core}} = c(1 - \alpha^+ - \beta^+)(1 - \alpha^- - \beta^-). \quad (10)$$

II. CONDITION FOR CORE PERCOLATION

It is easy to see that the core in a undirected network with degree distribution $P(k)$ is the very same as in a directed network with the same out- and in-degree distributions, i.e. $P^+(k) = P^-(k) = P(k)$. Therefore we can deal with directed network for generality. As n_{core} is a continuous function of α^\pm , we focus on α^\pm , which is the smallest root of the function $f^\pm(x) \equiv A^\pm(A^\mp(x)) - x$. There are several interesting facts about the function $f^\pm(x)$. First of all, since $A^\pm(x)$ is a monotonically decreasing function for $x \in [0, 1]$ and $A^\pm(0) = 1$ is the maximum (see Figs.2, 3), we have $f^\pm(0) > 0$ and $f^\pm(1) < 0$ (see Fig.3c,d). Consequently, the number of roots (with multiplicity) of $f^\pm(x)$ in $[0, 1]$ is odd, and numerical calculations suggest that this number is either 1 or 3 (see Figs.2, 3). Secondly, if $f^\pm(x_0) = 0$ then $f^\mp(A^\mp(x_0)) = 0$, which means $A^\mp(x)$ transforms the roots of $f^\pm(x)$ to the roots of $f^\mp(x)$. This also suggests that $f^\pm(x)$ *always* has a trivial root $\alpha^\pm = A^\pm(\alpha^\mp) = 1 - \beta^\pm$. (For undirected networks, $f(x)$ *always* has a trivial root $\alpha = A(\alpha) = 1 - \beta$.) Since $A^\mp(x)$ is a monotonically decreasing function and α^\pm is the smallest root of $f^\pm(x)$, $A^\mp(\alpha^\pm) = 1 - \beta^\mp$ is therefore the largest root of $f^\mp(x)$. Hence $1 - \beta^\pm - \alpha^\pm$ is the difference between the largest and the smallest roots of $f^\pm(x)$ (see Fig.2). Consequently, if $f^\pm(x)$ has only one root (which then must be the trivial root $\alpha^\pm = A^\pm(\alpha^\mp) = 1 - \beta^\pm$), then $1 - \beta^\pm - \alpha^\pm = 0$. According to Eq.8, this

implies that there is no core. On the other hand, if multiple roots exist and they are different then $1 - \beta^\pm - \alpha^\pm > 0$, and the core will develop.

We apply the above condition to the following random undirected networks with specific degree distributions[10]. (1) Erdős-Rényi (ER)[7, 8] networks with Poisson degree distribution $P(k) = e^{-c} c^k / k!$, $A(x) = e^{-cx}$ and $f(x) = \exp(-ce^{-cx}) - x$. As shown in Fig.3a, the core percolation occurs at $c = c^* = e$, which agrees with previous theoretical results[32, 33, 39]. (2) Exponentially distributed graphs with $P(k) = (1 - e^{-1/\kappa})e^{-k/\kappa}$ and mean degree $c = e^{-1/\kappa}/(1 - e^{-1/\kappa})$. We find that core percolation occurs at $c = c^* = 4$. (3) Purely power-law distributed networks with $P(k) = k^{-\gamma}/\zeta(\gamma)$ for $k \geq 1$, $\gamma > 2$ and $\zeta(\gamma)$ the Riemann ζ function. We find that $f(x)$ has no multiple roots and hence $n_{\text{core}} = 0$ for all $\gamma > 2$. In other words, for purely scale-free (SF) networks, the core does not exist. (4) Power-law distributed networks with exponential degree cutoff, i.e. $P(k) = \frac{k^{-\gamma} e^{-k/\kappa}}{\text{Li}_\gamma(e^{-1/\kappa})}$ for $k \geq 1$ with $\text{Li}_n(x)$ the n th polylogarithm of x . We find that $n_{\text{core}} = 0$ for $\gamma > \gamma_c(\kappa)$, and the threshold value $\gamma_c(\kappa)$ approaches 1 as κ increases. Hence, for SF networks with exponential degree cutoff the core still does not exist for all $\gamma > 1$. (5) Asymptotically SF networks generated by the static model with $P(k) = \frac{[\frac{\gamma}{2}(1-\xi)]^{1/\xi}}{\xi} \frac{\Gamma(k-1/\xi, \frac{\gamma}{2}[1-\xi])}{\Gamma(k+1)}$, where $\Gamma(s)$ is the gamma function and $\Gamma(s, x)$ the upper incomplete gamma function[44–46]. In the large k limit, $P(k) \sim k^{-(1+\frac{1}{\xi})} = k^{-\gamma}$ where $\gamma = 1 + \frac{1}{\xi} > 2$. For small k , $P(k)$ deviates significantly from the power-law distribution[45] and there are much fewer small-degree nodes than the purely scale-free networks, which results in a drastically different core percolation behavior.

Hereafter, we systematically study the net effect of adding more links (i.e. increasing mean degree c , yet without changing other parameters in $P(k)$) on core percolation. ER networks and the asymptotically SF networks generated by the static model naturally serve this purpose, since their mean-degree is an independent and explicit tuning parameter.

III. NATURE OF CORE PERCOLATION

We observed that if the mean degree c is small, then $f^\pm(x)$ has one root, but if c is large, $f^\pm(x)$ has three roots (see Figs.2, 3). At the critical point $c = c^*$, the number of roots jumps from 1 to 3 by the appearance of one new root with multiplicity 2. (Note that $f^\pm(x)$ cannot immediately intersect the x -axis at two new points, but it touches first.) This explains why the core percolation occurs at $c = c^*$.

According to the transformation from the roots of $f^\pm(x)$ to the roots of $f^\mp(x)$ through $A^\mp(x)$, for either $f^+(x)$ or $f^-(x)$ (depending on the details of $P^+(k)$ and $P^-(k)$) its new root at $c = c^*$

is smaller than its original root; and for either $f^-(x)$ or $f^+(x)$ the new root at $c = c^*$ is larger than the original root; or there is a degenerate case when this new root is the same as the original root for both $f^+(x)$ and $f^-(x)$. For example, for directed asymptotically SF networks generated by the static model with $\gamma_{\text{in}} = 2.7, \gamma_{\text{out}} = 3.0$, the new root (marked as green dot) of $f^+(x)$ at $c = c^*$ is smaller than the original root (green square) of $f^+(x)$ (see Fig.3c), and the new root (green square) of $f^-(x)$ at $c = c^*$ is larger than the original root (green circle) of $f^-(x)$ (see Fig.3d). In other words, at the critical point, for either $f^+(x)$ or $f^-(x)$, its smallest two roots are the same, and for the other function (either $f^-(x)$ or $f^+(x)$), its largest two roots are the same (see Fig.3c,d). While for directed networks with $P^+(k) = P^-(k) = P(k)$, i.e. the degenerate case, we have $f^+(x) = f^-(x) = f(x)$, and the new root of $f(x)$ at $c = c^*$ has to be the same as the original root of $f(x)$, i.e. all three roots must be the same (see Fig.3a). Therefore at the critical point, unless in the degenerate case, α^+ together with β^- (or α^- together with β^+) decrease discontinuously, which implies a discontinuous transition in the core size. To sum up, in the degenerate case that $P^+(k) = P^-(k) = P(k)$ core percolation is continuous, but for general non-degenerate case $P^+(k) \neq P^-(k)$, we have a discontinuous transition in both n_{core} and l_{core} . These results are clearly shown in Fig.3b,e.

At the critical point c^* , $f^\pm(x)$ touches the x -axis at its new root (see Fig.3c,d), hence we have either $f^+(\alpha^+) = (f^+)'(\alpha^+) = 0$ (or $f^-(1 - \beta^-) = (f^-)'(1 - \beta^-) = 0$), which enable us to calculate the core percolation threshold c^* . In the degenerate case, if $c \leq c^*$ then $f(\alpha) = f'(\alpha) = 0$ can be further simplified as $A(\alpha) = \alpha$ and $[A'(\alpha)]^2 = 1$. The results of c^* for ER and SF networks generated by the static model are shown in Fig.4a.

The discontinuity in n_{core} and l_{core} at c^* , denoted by Δ_n and Δ_l respectively, can also be calculated

$$\Delta_n = \frac{1}{2} (\Delta_n^+ + \Delta_n^-) \quad (11)$$

$$\Delta_l = c^*(1 - \beta^{-,*} - \alpha^{-,*})(1 - \beta^{+,*} - \alpha^{+,*}) \quad (12)$$

with $\Delta_n^\pm \equiv G^\pm(1 - \alpha^{\mp,*}) - G^\pm(\beta^{\mp,*}) - c^*(1 - \beta^{\mp,*} - \alpha^{\mp,*})\alpha^{\pm,*}$. The results of Δ_n for ER and SF networks generated by the static model are shown in Fig.4b. We find that $\Delta_n \rightarrow 0$ as $\gamma_{\text{in}} \rightarrow \gamma_{\text{out}}$, consistent with the result obtained above that core percolation is continuous for undirected networks or directed networks with $P^+(k) = P^-(k)$. We also find that Δ_n increases as the differences between γ_{in} and γ_{out} increases.

We can further show that in the general non-degenerate case, core percolation is actually a hybrid phase transition[12, 13, 47], i.e. n_{core} (or l_{core}) has a jump at the critical point as at a

first-order phase transition but also has a critical singularity as at a continuous transition. The results are summarized here: in the critical regime $\epsilon = c - c^* \rightarrow 0^+$

$$n_{\text{core}} - \Delta_n \sim (c - c^*)^\eta \quad (13)$$

$$l_{\text{core}} - \Delta_l \sim (c - c^*)^\theta \quad (14)$$

with the critical exponents $\eta = \theta = \frac{1}{2}$. Our calculations do not use any specific functional form of $A^\pm(x)$. Instead, we only assume that they are continuous functions of the mean degree c . Interestingly, in the degenerate or undirected case, one has a continuous phase transition ($\Delta_n = \Delta_l = 0$) but with a completely different set of critical exponents: $\eta' = \theta' = 1$ [32].

IV. NUMERICAL RESULTS

We check our analytical results with extensive numerical calculations by performing the GLR procedure on finite discrete networks generated by the static model[44–46]. Fig.5a and 5b show n_{core} and l_{core} (in symbols) for undirected ER networks and asymptotically SF networks with different degree exponents. For comparison, analytical results for infinite large networks are also shown (in lines). Clearly, core percolation is continuous in this case. This is fundamentally different from the $k \geq 3$ -core percolation, which becomes discontinuous for ER networks and SF networks with $\gamma > 3$ [11, 12].

Fig.5c and 5d show the results of n_{core} and l_{core} for directed networks. For directed networks with the same in- and out-degree distributions, e.g. directed ER networks or directed SF networks with $\gamma_{\text{in}} = \gamma_{\text{out}}$ generated by the static model, the core percolation is still continuous. But for directed networks with different in- and out-degree distributions, e.g. directed SF networks with $\gamma_{\text{in}} \neq \gamma_{\text{out}}$ generated by the static model, the core percolation looks discontinuous. The discontinuity in n_{core} (or l_{core}) increases as the difference between γ_{in} and γ_{out} increases (see Fig.5e,f).

V. REAL NETWORKS

We also apply our theory to real-world networks with known degree distributions. In Fig.6 we demonstrate that in some cases our analytical results calculated from Eqs.4, 5 (or Eqs.9, 10) with degree distribution as the only input predict with surprising accuracy the core size of real networks. Yet, in other cases there is a noticeable difference between theory and reality, which suggests the presence of extra structure in the real-world networks that is not captured by the

degree distribution. In particular we find that almost all the directed real-world networks have larger core sizes than the theoretical predictions (see Fig.6a,b). In other words, those networks are “overcored”. While if we treat those networks as undirected ones, their core sizes deviate from our theory in a more complicated manner. The effects of higher order correlations (e.g. degree correlations[48], clustering[49], loop structure[50] and modularity[51]) may play very important roles to explain the discrepancy between theory and reality.

VI. CONCLUSION

In sum, we analytically solve the core percolation problem in both undirected and directed random networks with arbitrary degree distributions. We show the condition for core percolation. We find it is continuous in undirected networks (if it occurs), while it becomes discontinuous or hybrid in directed networks unless the in- and out-degree distributions are the same. Within each case, the critical exponents associated with the critical singularity are universal for random networks with arbitrary degree distributions parameterized continuously in mean degree. But the two cases have totally different sets of critical exponents. These results vividly illustrate that core percolation is a fundamental structural transition in complex networks and its implication on other problems, e.g. conductor-insulator transitions, combinatorial optimization problems, and network controllability issue, deserves further exploration. The analytical framework presented here also raises a number of questions, answers to which would further improve our understanding of core percolation on complex real-world networks. For example, we focused on uncorrelated random networks and leave the systematic studies of the effects of higher order correlations as future work.

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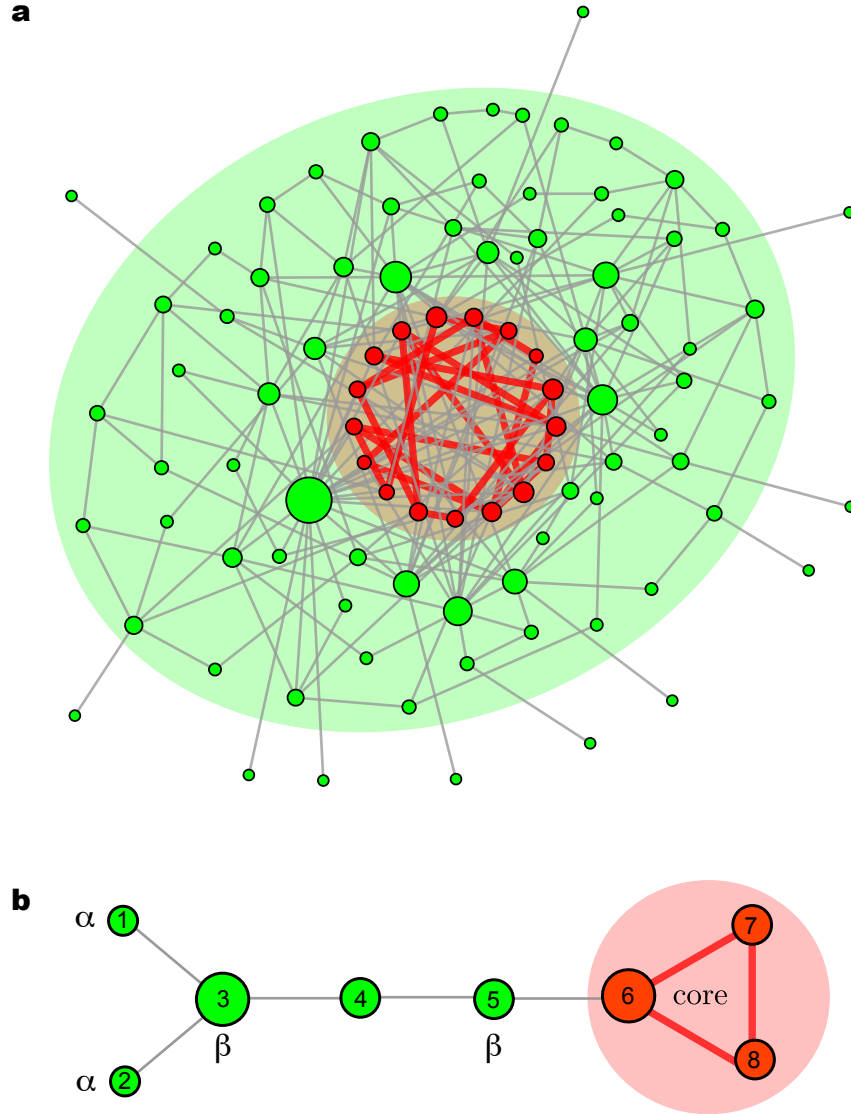


FIG. 1: **The core of a small network.** **a**, The core (highlighted in red) obtained after the greedy leaf removal procedure is fundamentally different from the 2-core (highlighted in green) obtained by iteratively removing nodes of degree less than 2. The 2-core contains the core, whereas the opposite is not true. Size of nodes are roughly proportional to the degree of nodes. **b**, Removal categories of nodes according to how they can be removed during the greedy leaf removal procedure. Red nodes are non-removable, i.e. they belong to the core. Green nodes are removable: nodes v_1 and v_2 are α -removable; nodes v_3 and v_5 are β -removable. White node v_4 is removable but it is neither α -removable nor β -removable. Node v_5 is β -removable because v_4 will become a leaf node after removing node v_1 (or v_2) together with v_3 .

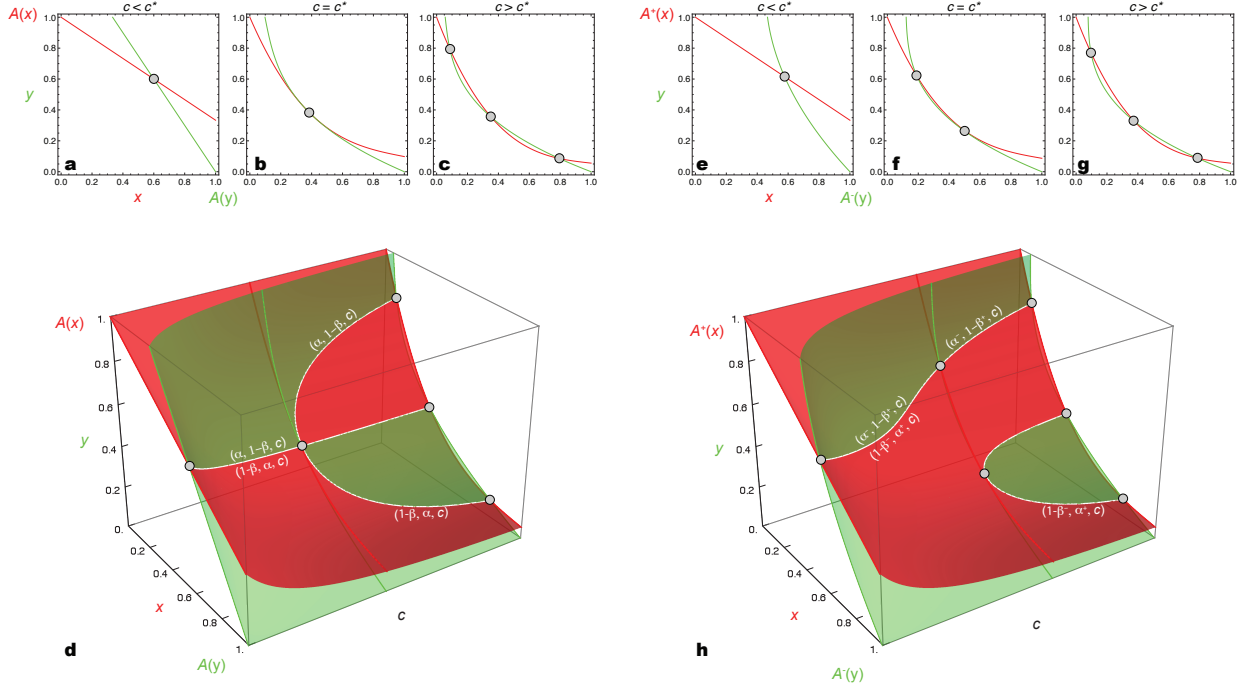


FIG. 2: Graphical solution of the self-consistent equations. **a-d**, For undirected networks, the function $A(x)$ transforms the roots of $f(x)$ to the roots of the same function $f(x)$. The graphical solution of $f(x) = A(A(x)) - x = 0$ is best illustrated by plotting the two curves $A(x)$ vs. x (in red) and y vs. $A(y)$ (in green) in the same coordinate system. The coordinates of the intersection point(s) of the two curves give the solution(s) of $f(x) = 0$. In **a**, **b**, and **c**, we show the graphical solutions for $c < , = ,$ and $> c^*$, respectively. **d**, By drawing the two curves ($A(x)$ vs. x) and (y vs. $A(y)$) at different mean degrees c , we get two surfaces. The intersection curve of the two surfaces yields the solutions of $f(x) = 0$ at different c values. For $c < c^*$, the intersection curve has one branch given by $(\alpha, 1 - \beta, c) = (1 - \beta, \alpha, c)$. For $c > c^*$, the intersection curve has three branches. The top and bottom branches are given by $(\alpha, 1 - \beta, c)$ and $(1 - \beta, \alpha, c)$, respectively. **e-h**, For directed networks, $A^\pm(x)$ transforms the roots of $f^\mp(x)$ to the roots of $f^\pm(x)$. The graphical solution of $f^\pm(x) = A^\pm(A^\mp(x)) - x = 0$ can be illustrated by plotting $A^+(x)$ vs. x (in red) and y vs. $A^-(y)$ (in green) in the same coordinate system. The x -coordinate (or y -coordinate) of the intersection point(s) of the two curves give the solution(s) of the equation $f^-(x) = 0$ (or $f^+(x) = 0$, respectively). In **e**, **f**, and **g**, we show the graphical solutions for $c < , = ,$ and $> c^*$, respectively. **h**, By drawing the two curves ($A^+(x)$ vs. x) and (y vs. $A^-(y)$) at different mean degrees c , we get two surfaces. The intersection curve of the two surfaces yields the solutions of $f^\pm(x) = 0$ at different c values. For $c < c^*$, the intersection curve has one branch given by $(\alpha^-, 1 - \beta^+, c) = (1 - \beta^-, \alpha^+, c)$. For $c > c^*$, the intersection curve has three branches. The top and bottom branches are given by $(\alpha^-, 1 - \beta^+, c)$ and $(1 - \beta^-, \alpha^+, c)$, respectively.

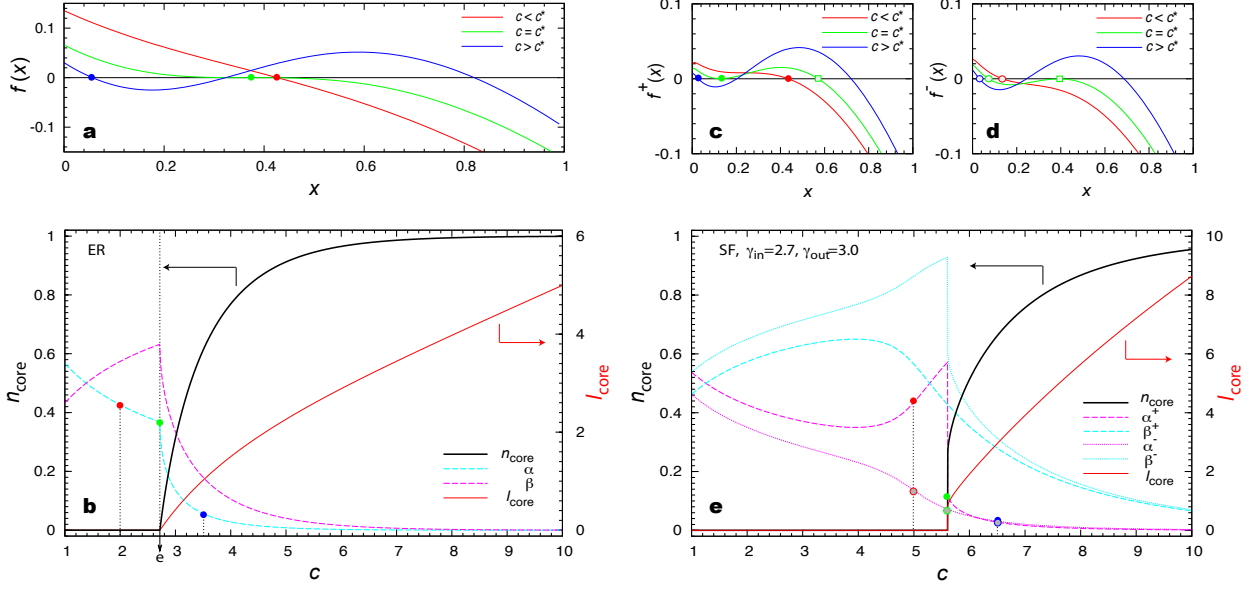


FIG. 3: **Analytical solution of the core percolation.** **a-b**, Undirected Erdős-Rényi (ER) random networks. **a**, α is the smallest root of the function $f(x) \equiv A(A(x)) - x$, represented by red, green, and blue dots for $c < , =$, and $> c^* = e$, respectively. **b**, $\alpha, \beta, n_{\text{core}}$ and l_{core} as functions of the mean degree c . **c-e**, Directed asymptotically scale-free (SF) random networks generated by the static model. Both the in-degree and out-degree distributions of the networks are scale-free with degree exponents $\gamma_{\text{in}} = 2.7$ and $\gamma_{\text{out}} = 3.0$. **c**, **d**, α^\pm is the smallest root of the function $f^\pm(x) \equiv A^\pm(A^\mp(x)) - x$, represented by red, green, and blue dots for $c < , =$, and $> c^* \simeq 11.2$, respectively. **e**, $\alpha^\pm, \beta^\pm, n_{\text{core}}$ and l_{core} as functions of the mean degree c . The jumps in α^+ and β^- result in the jumps in n_{core} and l_{core} , hence the first-order core percolation occurs.

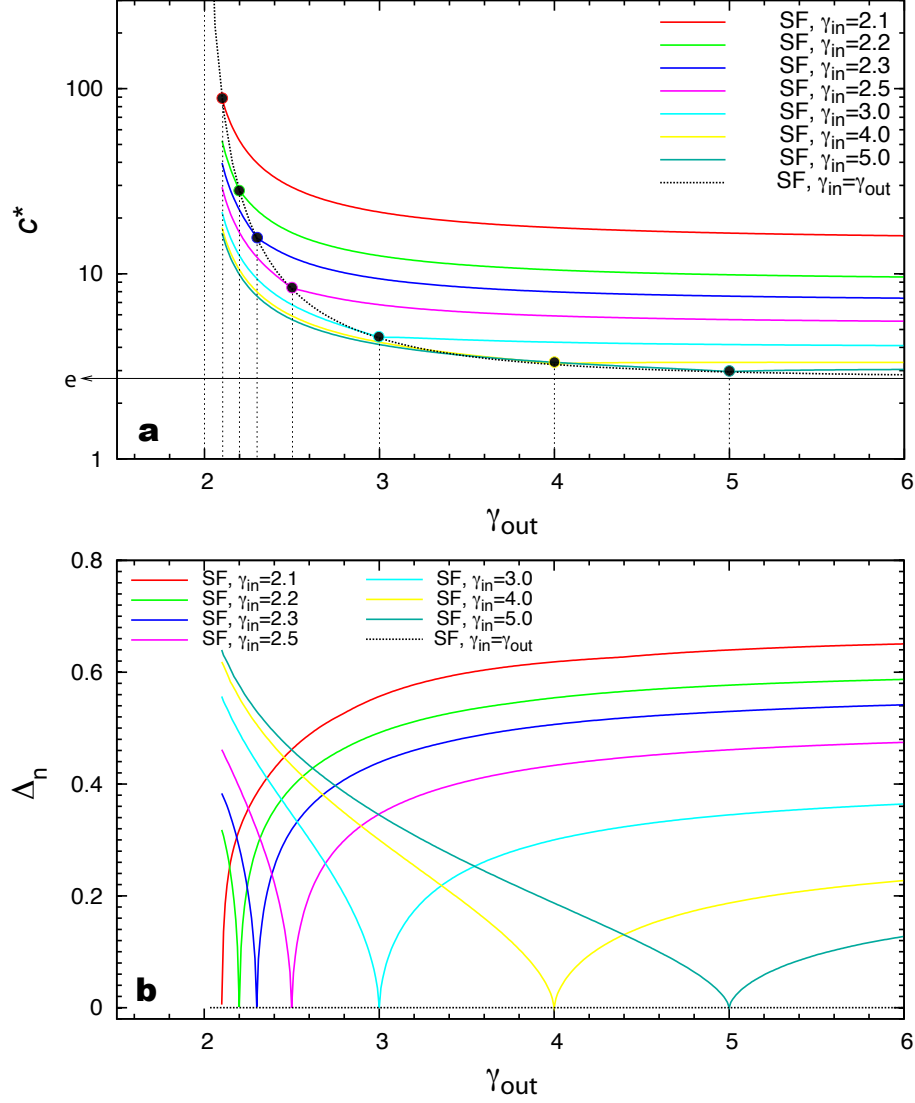


FIG. 4: Threshold and discontinuity of core percolation. **a**, Analytical solution of the core percolation threshold c^* calculated by solving $f^\pm(x) = f^{\pm'}(x) = 0$ for model networks. For ER networks, $c^* = e$. For undirected asymptotically SF networks generated by the static model, $c^* \rightarrow \infty$ as $\gamma \rightarrow 2$, and $c^* \rightarrow e$ as $\gamma \rightarrow \infty$. **b**, The discontinuity Δ_n in n_{core} at $c = c^*$ for model networks. For undirected or directed networks with $P^+(k) = P^-(k)$, $\Delta_n = 0$. For directed network, Δ_n increases as the difference between the in- and out-degree distributions (quantified by the difference between the degree exponents γ_{in} and γ_{out}) increases.

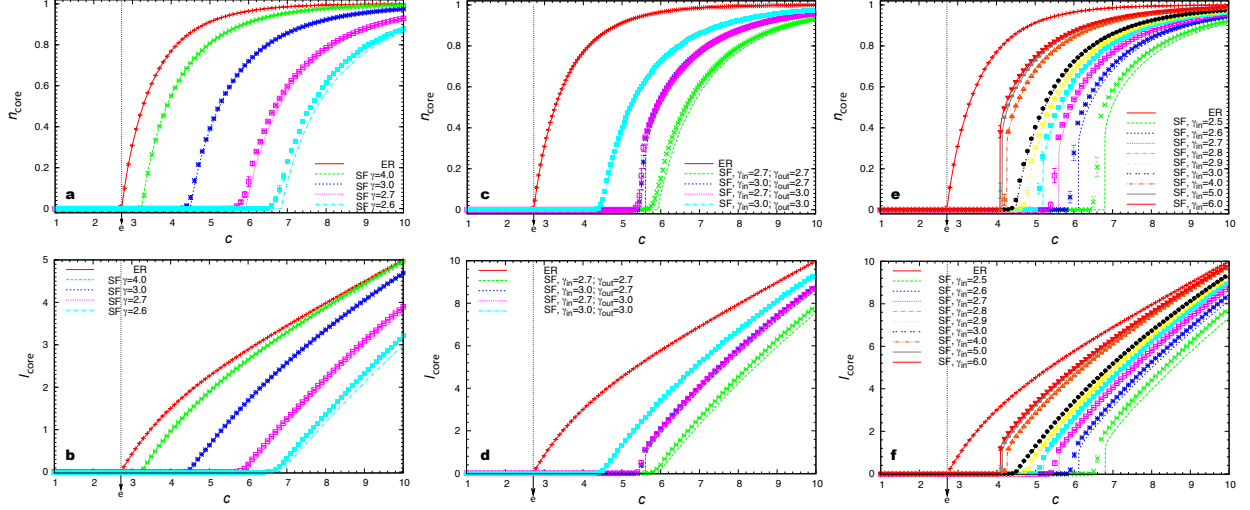


FIG. 5: Core percolation in random networks. Symbols are numerical results calculated from the GLR procedure on finite discrete networks constructed with the static model[44] with $N = 10^5$. The numerical results are averaged over 20 realizations with error bars defined as s.e.m. Lines are analytical results for infinite large system ($N \rightarrow \infty$) calculated from Eq.4 and 5 for undirected networks or Eq.9 and 10 for directed networks. Finite size effects are more discernable for $\gamma \rightarrow 2$, which can be eliminated by imposing degree cutoff in constructing the SF networks[52, 53]. **a-b**, The normalized core size ($n_{\text{core}} = N_{\text{core}}/N$) and the normalized number of edges in the core ($l_{\text{core}} = L_{\text{core}}/N$) for undirected model networks: Erdős-Rényi (ER) and asymptotically scale-free (SF) with different values of γ . For both model networks, the core percolation is continuous, which is fundamentally different from the $k \geq 3$ -core percolation, which becomes discontinuous for ER networks and SF networks with $\gamma > 3$ [11, 12]. **c-d**, n_{core} and l_{core} for directed ER and asymptotically SF model networks. The core percolation is continuous if the out- and in-degree distributions are the same ($P^+(k) = P^-(k)$) while it becomes discontinuous if $P^+(k) \neq P^-(k)$. **c-d**, For directed SF networks with fixed $\gamma_{\text{out}} = 3.0$, by tuning γ_{in} we see that the discontinuity in both n_{core} and l_{core} become larger as the difference between γ_{in} and γ_{out} increases.

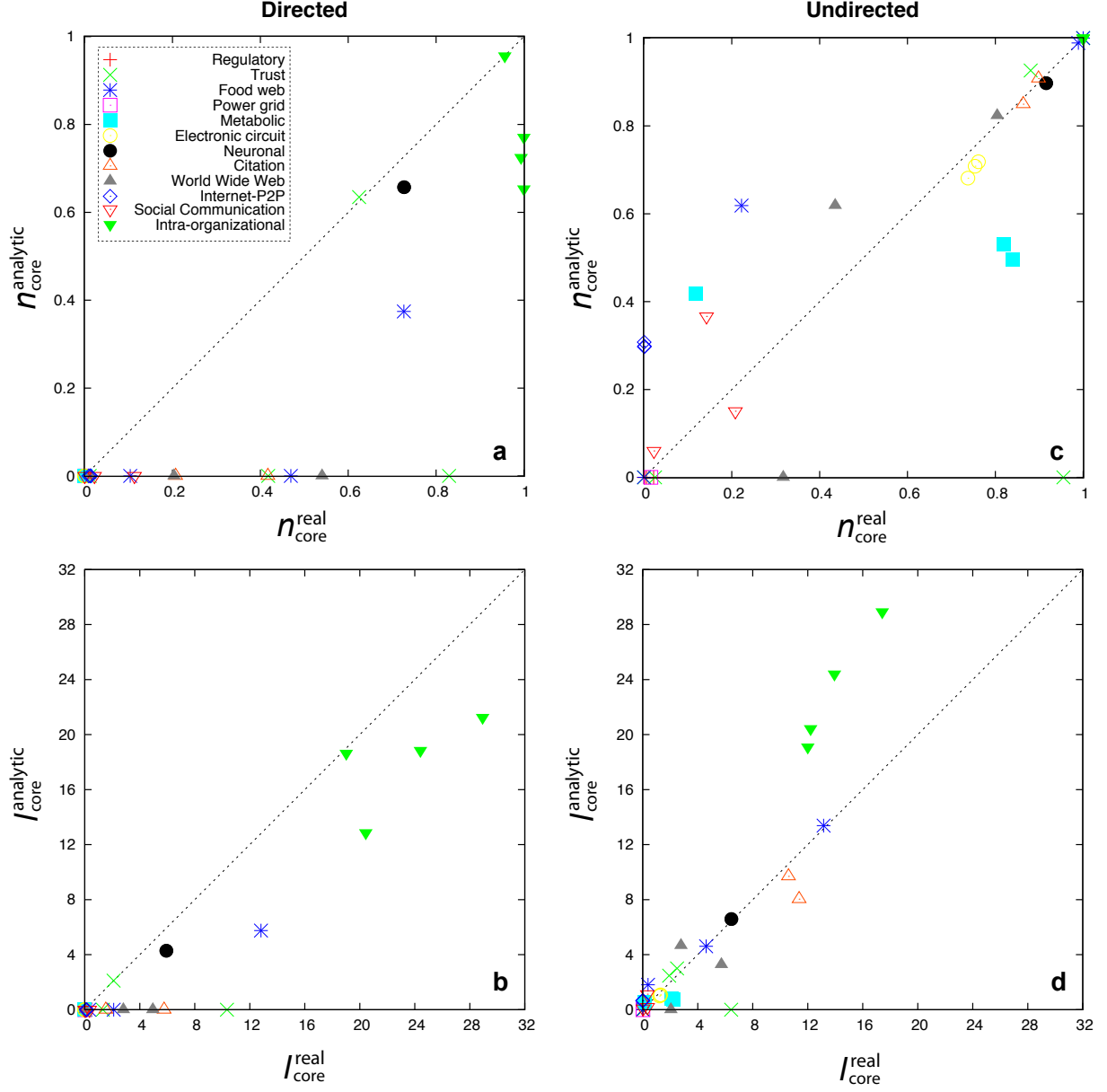


FIG. 6: **Normalized core size for real networks, compared with analytical predictions.** All the real networks considered here are directed. For data sources and references, see Ref.[27] Supplementary Information Sec.VI. **a-b**, By applying the GLR procedure we yield $n_{\text{core}}^{\text{real}}$ and $l_{\text{core}}^{\text{real}}$. Using Eq.9 and Eq.10 with out- and in-degree distributions ($P^+(k)$ and $P^-(k)$) as the only inputs, we obtain $n_{\text{core}}^{\text{analytic}}$ and $l_{\text{core}}^{\text{analytic}}$. **c-d** By ignoring the direction of the edges, we can treat the original directed networks as undirected ones and apply the GLR procedure to get $n_{\text{core}}^{\text{real}}$ and $l_{\text{core}}^{\text{real}}$. Similarly, we can calculate $n_{\text{core}}^{\text{analytic}}$ and $l_{\text{core}}^{\text{analytic}}$ by using Eq.4 and Eq.5 with the degree distribution $P(k)$ as the only input.

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